

MONOLITHIC MODULES OVER NOETHERIAN RINGS

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ABSTRACT. We study finiteness conditions on essential extensions of simple modules over the quantum plane, the quantized Weyl algebra and Noetherian down-up algebras. The results achieved improve the ones obtained in [5] for down-up algebras.

1. INTRODUCTION

In this paper we consider the following property of a Noetherian ring A :

(\diamond) Injective hulls of simple left A -modules are locally Artinian.

Property (\diamond) has an interesting history. Indeed it was shown by A.V. Jategaonkar [12] and J.E. Roseblade [20] that if G is a polycyclic-by-finite group, then the group ring RG has property (\diamond) whenever R is the ring of integers, or is a field that is algebraic over a finite field see also [18] Section 12.2. This result is the key step in the positive solution of a problem of P. Hall, [9]. P. Hall asked whether every finitely generated abelian-by-(polycyclic-by-finite) group is residually finite. In [20] a module M is called *monolithic* if it has a unique minimal submodule. Note that A has property (\diamond) if and only if every finitely generated monolithic A -module is Artinian. We have revived the older, shorter terminology in the title of this paper. A.V. Jategaonkar showed in [11] that a fully bounded Noetherian ring R satisfies property (\diamond), and used this fact to show that Jacobson's conjecture holds for R .

Returning to the group ring situation, suppose G is a polycyclic-by-finite group, K is a field, $A = KG$ and E is the injective hull of a finite-dimensional A -module. It was shown by K.A. Brown, [3] that if K has characteristic zero, then E is locally finite dimensional, and this fact and some Hopf algebra theory was used by S. Donkin to show that E is in fact Artinian [8]. Note that injective comodules over coalgebras are always locally finite dimensional. Similar results were obtained when K has positive characteristic by the second author [15] using methods that more closely follow the argument used for commutative rings in [21].

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The first examples of Noetherian rings for which property (\diamond) does not hold were given by the second author for group algebras and enveloping algebras, see [16], [17] and [6, Example 7.15]. On the other hand R.P. Dahlberg [7] showed that injective hulls of simple modules over $U(\mathfrak{sl}_2)$ are locally Artinian.

Interest in property (\diamond) was renewed recently by a question of P.F. Smith. Smith asked whether Noetherian down-up algebras have property (\diamond) . Given a field K and α, β, γ arbitrary elements of K , the associative algebra $A = A(\alpha, \beta, \gamma)$ over K with generators d, u and defining relations

$$(R1) \quad d^2u = \alpha dud + \beta ud^2 + \gamma d$$

$$(R2) \quad du^2 = \alpha udu + \beta u^2d + \gamma u$$

is called a down-up algebra. Down-up algebras were introduced by G. Benkart and T. Roby [1]. In [13] it is shown that $A(\alpha, \beta, \gamma)$ is Noetherian if and only if $\beta \neq 0$. Some examples of down-up algebras with property (\diamond) were given in [5]. In this paper we study Noetherian down-up algebras having property (\diamond) , and in particular we exhibit the first examples that do not have this property. These examples are constructed using the fact that when $\gamma = 0$, (resp. $\gamma = 1$) the quantum plane, (resp. the quantized Weyl algebra) is an image of A .

An interesting class of down-up algebras arises in the following way. For $\eta \neq 0$, let A_η be the algebra with generators h, e, f and relations

$$he - eh = e,$$

$$hf - fh = -f,$$

$$ef - \eta fe = h.$$

Then A_η is isomorphic to a down-up algebra $A(1 + \eta, -\eta, 1)$ and conversely any down-up algebra $A(\alpha, \beta, \gamma)$ with $\beta \neq 0 \neq \gamma$ and $\alpha + \beta = 1$ has the above form. Note that $A_1 \simeq U(\mathfrak{sl}(2))$ and $A_{-1} \simeq U(\mathfrak{osp}(1, 2))$. When η is not a root of unity, we have been unable to determine whether property (\diamond) holds. However we resolve the issue in all other cases. Our main result is as follows.

Theorem 1.1. *Suppose that $A = A(\alpha, \beta, \gamma)$ is a Noetherian down-up algebra, and assume that if $\alpha + \beta = 1$, and $\gamma \neq 0$, then β is a root of unity. Then any finitely generated monolithic A -module is Artinian if and only if the roots of $X^2 - \alpha X - \beta$ are roots of unity.*

We remark that a characterization of property (\diamond) for Noetherian rings remains rather elusive. Even a comparison of the examples for the quantum plane and quantized Weyl algebra does not seem easy to make, see Section 4 for further remarks. Thus it seems worthwhile to study examples of rings with low GK-dimension, and down-up algebras provide an interesting test-case for property (\diamond) . Much current research in non-commutative algebraic geometry also centers on low dimensional

algebras, and in particular down-up algebras are studied as non-commutative three-folds by Kulkarni in [14].

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2. PRELIMINARIES.

If $r \in K$ and x, y are elements of a K -algebra we set $[x, y]_r = xy - r yx$. Throughout this paper we will assume the equation $0 = \lambda^2 - \alpha\lambda - \beta$ has roots $r, s \in K$. Suppose $q \in K$ is nonzero and consider the algebra $B(q) = K[a, b]$ generated by a, b subject to the relation $ab = qba$. In addition let $C(q) = K[a, b]$ denote the algebra generated by a, b subject to the relation $ab - qba = 1$. The algebras $B(q), C(q)$ are known as the *coordinate algebra of the quantum plane* and the *quantized Weyl algebra* respectively.

Lemma 2.1.

- (a) *The algebra $B(r)$ is a homomorphic image of $A = A(\alpha, \beta, 0)$.*
- (b) *If $s \neq 1$ the algebra $C(r)$ is a homomorphic image of $A = A(\alpha, \beta, 1)$.*

Proof. If $\gamma = 0$, relations (R1) and (R2) can be written in the form

$$[d, [d, u]_r]_s = [[d, u]_r, u]_s = 0.$$

Thus both relations follow from the relation $[d, u]_r = 0$, so there is a map from $A = A(\alpha, \beta, \gamma)$ onto $B(r)$ sending d to a and u to b .

On the other hand if $\gamma \neq 0$, we can assume $\gamma = 1$. If $s \neq 1$, let $t \in K$ be such that $t(s - 1) = 1$. Relations (R1) and (R2) can now be written in the form

$$[d, [d, u]_r - t]_s = [[d, u]_r - t, u]_s = 0.$$

Since $[ta, b]_r - t = 0$ in $C(r)$, there is an homomorphism from A onto $C(r)$ sending d to ta and u to b . \square

The above Lemma will be used, together with the results of the next two subsections, to produce examples of down-up algebras that do not satisfy property (\diamond) . Note however that if exactly one of the roots of the Equation $X^2 - \alpha X - \beta$ is equal to 1, the Lemma tells us only that the first Weyl algebra is a homomorphic image of $A = A(\alpha, \beta, 1)$. In this case the Lemma is of no use in constructing counterexamples.

3. THE COORDINATE RING OF THE QUANTUM PLANE.

If q is an element of K which is not a root of unity we show that $B = B(q)$ does not satisfy property (\diamond) . Consider the left ideals $I = B(ab - 1)(a - 1) \subset J = B(a - 1)$, and set $M = B/I, V = J/I$ and $W = B/J$. Then there is an exact sequence

$$0 \longrightarrow V \longrightarrow M \longrightarrow W \longrightarrow 0.$$

Theorem 3.1.

- (a) *The module M is a non-Artinian essential extension of the simple submodule V .*
- (b) *The submodules of W are linearly ordered by inclusion, and are pairwise non-isomorphic.*

Proof. *Step 1: V is simple.* Clearly V is generated by the element $v_0 = (a-1) + I$. For $n \geq 0$, set

$$v_n = b^n v_0, \quad v_{-n} = a^n v_0.$$

Then using $abv_0 = v_0$, we obtain for all $n \geq 0$,

$$av_{n+1} = q^n v_n, \quad bv_{-n-1} = q^{-n-1} v_{-n}. \quad (1)$$

Furthermore for all integers n ,

$$abv_n = q^n v_n. \quad (2)$$

It is easy to see that V is spanned by the set $X = \{v_n | n \in \mathbb{Z}\}$, and it follows from equation (2) that the set X is linearly independent. Equation (2) also implies that any submodule of V is spanned by a subset of X . Then simplicity of V follows from equation (1).

Step 2: Proof of (b). Clearly W is generated by the element $w_0 = 1 + J$ and spanned over K by the set $Y = \{w_n | n \geq 0\}$, where $w_n = b^n w_0$. Furthermore for all $n \geq 0$,

$$aw_n = q^n w_n. \quad (3)$$

As in the proof of Step 1, Y is linearly independent. Equation (3) also implies that any submodule of W is spanned by a subset of Y . Now for all $n \geq 0$ set

$$W_n = \text{span}\{w_m | m \geq n\} = Bw_n.$$

Consideration of the action of b now shows that a complete list of non-zero submodules of W is

$$W = W_0 \supset W_1 \supset W_2 \dots$$

To complete the proof of (b) we observe that a acts as multiplication by q^n on the unique simple quotient of W_n .

Step 3: There is no element $v \in V$ such that $(a - q^m)v = v_m$. If $v \in V$ is non-zero we can write v as a linear combination of basis elements, $v = \sum_{i=r}^s \lambda_i v_i$, where λ_r, λ_s are nonzero. Then we set $|v| = s - r$. From equations (1), it follows that $|(a - q^m)v| = s - r + 1$. Clearly this gives the assertion.

Step 4: Proof of (a). Set $m_n = b^n + I$ for $n \geq 0$. Then m_n maps onto w_n under the natural map $M \rightarrow W$. Thus the set $\{v_n, m_p | n, p \in \mathbb{Z}, n \geq 0\}$ is a basis for M . Since $am_0 = m_0 + v_0$, it follows that

$$\begin{aligned} am_n &= q^n b^n am_0 \\ &= q^n (m_n + v_n). \end{aligned}$$

Suppose that $m = \sum_{i \in I} \lambda_i m_i + v$ is a nonzero element of M . We assume that $v \in V$, $|I|$ is non-empty, and that λ_i is a non-zero scalar for all $i \in I$. Then we show by induction on $|I|$ that $Bm \cap V$ is non-zero. Suppose that $n \in I$, and without loss that $\lambda_n = 1$. If $|I| = 1$, then $Bm \cap V$ contains

$$(a - q^n)(m_n + v) = q^n v_n + (a - q^n)v,$$

and by Step 3, this is non-zero. Similarly if $|I| > 1$, then Bm contains $(a - q^n)m$ and we have $(a - q^n)m = \sum_{j \in J} \mu_j m_j + v'$ with $J = I \setminus \{n\}$, $v' \in V$, and $\mu_j \neq 0$ for $j \in J$. Thus the result follows by induction. \square

4. THE QUANTIZED WEYL ALGEBRA

Throughout this section assume that q is an element of K which is not a root of unity. We show that the quantized Weyl algebra $C = C(q)$ does not have property (\diamond) . We begin with some comments which may serve to motivate our construction. Observe that in Theorem 3.1, the submodules of $W = Bw_0$ have the form $Bn^k w_0$ for some normal element n of B . An analogous statement holds for the Example from [6] mentioned in the Introduction. Now the element $n = ab - ba \in C$ is normal, and we can in fact repeat this strategy. Note however that n has degree two with respect to a natural filtration on C , whereas in the earlier examples the normal element had degree one. For this reason, we have not attempted to give a more unified treatment of our results.

It is reasonable to look for a C -module W such that $W = K[n]$ as a $K[n]$ -module with (n^i) a submodule of W for each i . Note that $\bar{C} = C/Cn \simeq K[a^{\pm 1}]$, and that if such a module W exists, then each factor $(n^i)/((n^{i+1}))$ is a one-dimensional \bar{C} -module. Based on these considerations, it is not hard to determine the possibilities for W , and with a little experimentation, arrive at the required nonartinian monolithic module.

Consider the K -vector space M with basis $\{v_i, w_i : i, j \in \mathbb{N}\}$, and let $V = \text{span}_K \{v_i : i \in \mathbb{N}\}$, $W = M/V$. Define linear operators a and b on V by

$$av_0 = 0 \tag{4}$$

$$av_n = \frac{q^n - 1}{q - 1} v_{n-1} \tag{5}$$

$$bv_n = v_{n+1} \quad (6)$$

Next extend the action of a and b to M by setting

$$aw_n = q^n(w_n + w_{n+1}) \quad (7)$$

and

$$bw_n = \frac{q^{-n}}{1-q}w_n + (-1)^n v_0. \quad (8)$$

We then have

$$(ab - ba)w_n = -\frac{1}{q}w_{n+1}, \quad (9)$$

$$(ab - qba)w_n = w_n \quad (10)$$

It is now easy to see that M is a C -module, and V is a submodule of M .

Lemma 4.1. *The C -module V is simple.*

Proof. Since any element of V is of the form $v = a_0v_0 + a_1v_1 + \dots + a_nv_n$ for some $a_i \in K$, by equation (5) we deduce that $v_0 \in Cv$ for any nonzero $v \in V$. Hence V is simple and also $V = Cv_0$. \square

Theorem 4.2.

- (a) *The module M is a non-Artinian essential extension of the simple submodule V .*
- (b) *The submodules of W are linearly ordered by inclusion, and are pairwise non-isomorphic.*

Proof. First we prove (b). By equation (8) any submodule of W is spanned by a subset of $\{w_n : n \in \mathbb{N}_0\}$. For any $n \in \mathbb{N}$ set $W_n = \text{span}\{w_m : m \geq n\}$. Consideration on the actions of a and b shows that the complete list of non-zero submodules of W is

$$W = W_0 \supset W_1 \supset W_2 \supset \dots$$

Since b acts as multiplication by $\frac{q^{-n}}{1-q}$ on the unique simple quotient of W_n , the proof of (b) is complete.

Next we prove (a). By Lemma 4.1, V is simple and by (b) M is not Artinian. The rest of the proof consists of three steps.

- (i) Given $n \in \mathbb{N}$, by (8),

$$\left(b - \frac{q^{-n}}{1-q}\right)w_n = (-1)^n v_0 \in V \cap Cw_n, \quad (11)$$

so $Cw_n \cap V \neq 0$.

(ii) For any $n \in \mathbb{N}$, $C(w_n + v) \cap V \neq 0$. Indeed

$$(b - \frac{q^{-n}}{1-q})(w_n + v) = (-1)^n v_0 + (b - \frac{q^{-n}}{1-q})v. \quad (12)$$

So we must show that we can not have $v \in V \setminus \{0\}$ such that

$$\left(b - \frac{q^{-n}}{1-q}\right)v = (-1)^{n+1}v_0.$$

This follows since if $v = \lambda_0 v_0 + \dots + \lambda_m v_m$, for some $\lambda_0, \dots, \lambda_m \in K$ with $\lambda_m \neq 0$, then the coefficient of v_{m+1} in $(b - \frac{q^{-n}}{1-q})v$ is non-zero.

(iii) Let $m \in M \setminus V$. We show that $Cm \cap V \neq 0$. This will complete the proof. Without loss of generality we can write $m = w_n + \lambda_{n-1}w_{n-1} + \dots + \lambda_0 w_0 + v$ for some $v \in V$ and $\lambda_0, \dots, \lambda_{n-1} \in K$. Then $(b - \frac{q^{-n}}{1-q})m$ is a linear combination of w_{n-1}, \dots, w_0 , and the v_i with $i \in \mathbb{N}$. Either we are in case (i) or (ii) or if not, we apply $(b - \frac{q^{-k}}{1-q})$ for a suitable k and repeat the process. \square

5. A POSITIVE RESULT.

Let $A = A(\alpha, \beta, \gamma)$ be a down-up algebra and set $f(x) = x^2 - \alpha x - \beta$. Suppose that $f(x) = (x - r)^2$ where r is a primitive n^{th} root of unity. Thus $\alpha = 2r$ and $\beta = -r^2$. The goal of this section is to prove

Theorem 5.1. *A finitely generated essential extension of a simple A -module is Artinian.*

Suppose first that $\text{char}(K) = p$, and let $Z' = [d^{np}, u^{np}, (du - rud + \frac{\gamma}{r-1})^n]$. Using [10, Theorem 4.4] and [22, Lemma 2.2], it is easy to see that A is finitely generated over the central subalgebra Z' . Therefore A is PI and property (\diamond) holds. For the rest of this section we assume that $\text{char}(K) = 0$.

We denote the Krull dimension of a ring B by $\text{K.dim } B$. If $r = \gamma = 1$, then A is isomorphic to the enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$, and Theorem 5.1 holds by [7]. The proof depends on the fact that $\text{K.dim } A = 2$, and does not immediately adapt to our situation. A key step in our proof is the fact that a certain localization of A has Krull dimension 2, see Proposition 5.5.

We establish some preliminaries. By [5, Corollary 3.2] we may assume that $r \neq 1$. Hence case 3 of [4, §1.4] holds and we set

$$w_1 = (2\beta + \alpha)ud + (\alpha - 2)du + 2\gamma;$$

$$w_2 = 2du - 2ud$$

so that $\sigma(w_1) = rw_1$ and $\sigma(w_2) = rw_2 + w_1$. Set $w = w_1/2(r-1) = -rud + du + \varepsilon$ where $\varepsilon = \gamma/(r-1)$.

Lemma 5.2. $\overline{A} = A/Aw$ is a PI algebra.

Proof. Denote the images of u and d in \overline{A} by $\overline{u}, \overline{d}$, respectively. Then \overline{A} is generated by $\overline{u}, \overline{d}$ and we have that

$$-r\overline{u}\overline{d} + \overline{d}\overline{u} + \varepsilon = 0.$$

It follows that \overline{A} is isomorphic to a quantized Weyl algebra if $\gamma \neq 0$ and to the coordinate ring of a quantum plane if $\gamma = 0$. Since r is a primitive n^{th} root of unity for $n > 1$, it is well known that these algebras are PI. \square

Recall that given a ring D , an automorphism σ of D and a central element $a \in D$, the generalized Weyl algebra $D(\sigma, a)$ is the ring extension of D generated by x and y , subject to the relations: $xb = \sigma(b)x, by = y\sigma(b)$, for all $b \in D, yx = a, xy = \sigma(a)$. Noetherian down-up algebras can be presented as generalized Weyl algebras, see [13].

We need the following result of Bavula and van Oystaeyen [2, Theorem 1.2].

Theorem 5.3. *Let R be a commutative Noetherian ring with $\text{K.dim } R = m$ and let $T = R(\sigma, a)$ be a generalized Weyl algebra. Then $\text{K.dim } T = m$ unless there is a height m maximal ideal P of R such that one of the following holds:*

- a) $\sigma^n(P) = P$, for some $n > 0$;
- b) $a \in \sigma^n(P)$ for infinitely many n .

If there is an ideal P as above such that a) or b) holds then $\text{K.dim } T = m + 1$.

Given $\lambda_0, \lambda_1 \in K$ and $n \in \mathbb{Z}$ there is a unique $\lambda_n \in K$ such that

$$\lambda_n = \alpha\lambda_{n-1} + \beta\lambda_{n-2} + \gamma.$$

For all $n \in \mathbb{Z}$ we have, see [4, Lemma 2.3]

$$\sigma^{-n}(x - \lambda_0) = (x - \lambda_n, y - \lambda_{n+1}).$$

Lemma 5.4. *If M is a maximal ideal of R such that $x \in \sigma^n(M)$ for infinitely many n , then $\sigma^n(M) = M$ for some $n > 0$.*

Proof. We can assume that $x \in M$, that is $M = (x - \lambda_0, y - \lambda_1)$ with $\lambda_0 = 0$. The solution to the recursive relation is then given by

$$\lambda_n = c_1(r^n - 1) + c_2nr^n$$

for some fixed $c_1, c_2 \in K$. If $\lambda_n = 0$, then $nc_2 = c_1(1 - r^{-n})$, but the right side of this equation can take only finitely many values. Hence $c_2 = 0$ and the sequence $\{\lambda_n\}$ is periodic. Clearly this gives the result. \square

Since w is a normal element of A , the set $\{w^n | n \geq 0\}$ satisfies the Ore condition. We denote by A_w, R_w the localizations of A and R with respect to this set.

Proposition 5.5. $\text{K.dim } A_w = 2$.

Proof. Note that $A_w = R_w(\sigma, \lambda)$ is a generalized Weyl algebra, so by Lemma 5.4 and Theorem 5.3, we need to show that for any maximal ideal P of R_w and $n > 0$ we have $\sigma^n(P) \neq P$. We show that equivalently if M is a maximal ideal of R such that $\sigma^n(M) = M$, then $w \in M$. Indeed if $M = (w_1 - a_1, w_2 - a_2)$ then from [4, Lemma 2.2(ii)] we have $a_1 = 0$ and the result follows. \square

Proof of Theorem 5.1. Let V be a simple A -module and M a finitely generated essential extension of V . There are two cases.

If $wV = 0$, it is enough to show that $N = \text{ann}_M(Aw)$ is Artinian. However N is a module over the PI algebra A/Aw .

If $wV \neq 0$ then since w^n is central there exists $\lambda \in K$, $\lambda \neq 0$ such that $(w^n - \lambda)V = 0$. By [19, Theorem 3.15] $P = (w^n - \lambda)A$ is prime. By a similar argument as before we can assume $PM = 0$. Let $r, s \in K[w]$ be such that

$$1 = rw + s(w^n - \lambda).$$

This implies that $M = wM$ and $\text{ann}_M(w) = 0$, otherwise $wV = 0$. So M is an A_w -module which is annihilated by P_w . Since $\text{K.dim } A_w = 2$ and P_w is a nonzero prime ideal, A_w/P_w is a prime of Krull dimension one and the result follows from [16, Prop 5.5]. \square

6. DOWN-UP ALGEBRAS

Proof of Theorem 1.1 If the roots of $X^2 - \alpha X - \beta$ are both equal to one or distinct roots of unity it follows from [5, Corollary 3.2] that any finitely generated monolithic A -module is Artinian. By Theorem 5.1, the same holds if both roots of the quadratic equation are equal roots of unity.

Suppose that the roots of $X^2 - \alpha X - \beta$ are not both roots of unity. Note that 1 is a root of this equation, and in this case the other root is $-\beta$. By Lemma 2.1, either the coordinate algebra of the quantum plane $B(q)$ or the quantized Weyl algebra $C(q)$ (with q not a root of 1) is a homomorphic image of A depending on $\gamma = 0$ or $\gamma \neq 0$ respectively. Hence by Theorems 3.1 and 4.2 it follows that A does not satisfy condition (\diamond) . \square

REFERENCES

- [1] G. Benkart, T. Roby, Down-up algebras, J. of Algebra 209 (1998), 305–344. Addendum J. of Algebra 213 (1999), 378.
- [2] V. Bavula, F. Van Oystaeyen, Krull dimension of generalized Weyl algebras and iterated skew polynomial rings: commutative coefficients, J. Algebra 208 (1998) 1–34.
- [3] K.A. Brown, The structure of modules over polycyclic groups, Math. Proc. Cambridge Philos. Soc. 89, no. 2 (1981) 257–283.
- [4] P.A.A.B. Carvalho, I.M. Musson, Down-Up Algebras and their Representation Theory, J. Algebra 228 (2000) 286–310.

- [5] P.A.A.B. Carvalho, C. Lomp, D. Pusat-Yilmaz, Injective Modules over Down-Up Algebras, Glasgow Math. J. (to appear).
- [6] A.W. Chatters, C.R. Hajarnavis, Rings with chain conditions, Research Notes in Mathematics, Vol. 44, Pitman Advanced Publishing Program (1980).
- [7] R.P. Dahlberg, Injective hulls of simple $\mathfrak{sl}(2, \mathbb{C})$ modules are locally Artinian, Proc. Amer. Math. Soc. 107, no. 1 (1989) 35–37.
- [8] S. Donkin, On the Noetherian property in endomorphism rings of certain comodules, J. Algebra 70, no. 2 (1981) 394–419.
- [9] P. Hall, On the finiteness of certain soluble groups, Proc. London Math. Soc. (3) 9 (1959) 595–622.
- [10] J. Hildebrand, Centers of down-up algebras over fields of prime characteristic, Comm. Algebra 30 (2002) 171–191.
- [11] A.V. Jategaonkar, Jacobson’s conjecture and modules over fully bounded noetherian rings, J. Algebra 30 (1974) 103–121.
- [12] A.V. Jategaonkar, Integral group rings of polycyclic-by-finite groups, J. Pure Appl. Algebra 4 (1974) 337–343.
- [13] E. Kirkman, I.M. Musson, D.S. Passman, Noetherian down-up algebras, Proc. Amer. Math. Soc. 127 (11) (1999) 3161–3167.
- [14] R.S. Kulkarni, Down-up algebras at roots of unity, Proc. Amer. Math. Soc. 136(10) (2008) 3375–3382.
- [15] I.M. Musson, Injective modules for group rings of polycyclic groups, I, Quart. J. Math. Oxford Ser. (2) 31 (1980), 429–448.
- [16] I.M. Musson, Injective modules for group rings of polycyclic groups, II, Quart. J. Math. Oxford Ser. (2), 31 (1980) 449–466.
- [17] I.M. Musson, Some examples of modules over noetherian rings, Glasgow Math. J. 23 (1982) 9–13.
- [18] D.S. Passman, The algebraic structure of group rings, Reprint of the 1977 original, Robert E. Krieger Publishing Co. Inc., Melbourne, FL, (1985).
- [19] I. Praton, Primitive ideals of Noetherian down-up algebras, Comm. Algebra 32 (2004) 443–471.
- [20] J.E. Roseblade, Applications of the Artin-Rees lemma to group rings, Symposia Mathematica, Vol. XVII (Convegno sui Gruppi Infiniti, INDAM, Rome, 1973), Academic Press, London, 1976, 471–478.
- [21] D.W. Sharpe, P. Vamos, Injective modules, Cambridge Tracts in Mathematics and mathematical Physics, Vol. 62, Cambridge (1972).
- [22] K. Zhao, Centers of Down-Up algebras, J. Algebra 214 (1999) 103–121.

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